

Coloring triple systems with local conditions

Dhruv Mubayi*

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Abstract

We produce an edge-coloring of the complete 3-uniform hypergraph on n vertices with $e^{O(\sqrt{\log \log n})}$ colors such that the edges spanned by every set of five vertices receive at least three distinct colors. This answers the first open case of a question of Conlon-Fox-Lee-Sudakov [1] who asked whether such a coloring exists with $(\log n)^{o(1)}$ colors.

1 Introduction

A k -uniform hypergraph H (k -graph for short) with vertex set $V(H)$ is a collection of k -element subsets of $V(H)$. Write K_n^k for the complete k -graph with vertex set of size n . A (p, q) -coloring of K_n^k is an edge-coloring of K_n^k that gives every copy of K_p^k at least q colors. Let $f_k(n, p, q)$ be the minimum number of colors in a (p, q) -coloring of K_n^k . This paper deals only with $k = 3$.

Conlon-Fox-Lee-Sudakov [1] asked whether $f_3(n, p, p-2) = (\log n)^{o(1)}$ for $p \geq 3$ (the case $p = 4$ is easy). In this note we answer the first open case with a substantially smaller bound.

Theorem 1.

$$f_3(n, 5, 3) = e^{O(\sqrt{\log \log n})}.$$

The problem of determining $f_k(n, p, q)$ for fixed k, p, q has a long history, beginning with its introduction by Erdős and Shelah [3, 4], and subsequent investigation (for graphs) by Erdős and Gyárfás [5]. Studying $f_k(n, p, q)$ when $q = 2$ is equivalent to studying classical Ramsey numbers, and most of the effort on these problems has therefore been for $q > 2$. The simplest nontrivial case in this regime is $f_2(n, 4, 3)$, which was shown to be $n^{o(1)}$ in [10] and later $\Omega(\log n)$ (see [7, 9]). The same upper bound was shown for $f(n, 5, 4)$ in [6]. Conlon-Fox-Lee-Sudakov [2] recently extended this construction considerably by proving that $f_2(n, p, p-1) = n^{o(1)}$ for all fixed $p \geq 4$. Their result is sharp in the sense that $f_2(n, p, p) = \Omega(n^{1/(p-2)})$.

The first nontrivial hypergraph case is $f_3(n, 4, 3)$ and has tight connections to Shelah's breakthrough proof [12] of primitive recursive bounds for the Hales-Jewett numbers. Answering a question of

*Department of Mathematics, Statistics, and Computer Science, University of Illinois, Chicago, IL, 60607 USA. Research partially supported by NSF grant DMS-1300138. Email: mubayi@uic.edu

Graham-Rothschild-Spencer [8], Conlon et. al. [1] recently proved that $f_3(n, 4, 3) = n^{o(1)}$. They also posed a variety of basic questions about $f_3(n, p, q)$, including the one we address in this note.

Our construction uses an extension of the coloring in [10] together with the stepping up technique of Erdős and Hajnal. It is quite possible that, similar to the situation for graphs, other hypergraph cases will eventually be addressed by the ideas introduced here.

2 The Construction

We begin by defining an edge-coloring σ of the complete graph K_n whose vertices are ordered.

Construction of σ : Given integers $t < m$ and $n = \binom{m}{t}$, let $V(K_n)$ be the set of 0/1 vectors of length m with exactly t 1's. Write $v = (v(1), \dots, v(m))$ for a vertex. The vertices are naturally ordered by the integer they represent in binary, so $v < w$ iff $v(i) = 0$ and $w(i) = 1$ where i is the first position (minimum integer) in which v and w differ. By considering vertices as characteristic vectors of sets, we may assume that $V(K_n) = \binom{[m]}{t}$ whenever convenient. For each $B \in \binom{[m]}{t}$, let $f_B : 2^B \rightarrow [2^t]$ be a bijection. Given vectors $v < w$ that are characteristic vectors of sets $S < T$, let $c_1(vw) = \min\{i : v(i) = 0, w(i) = 1\}$, $c_2(vw) = \min\{j : j > i, v(i) = 1, w(i) = 0\}$, $c_3(vw) = f_S(S \cap T)$ and $c_4(vw) = f_T(S \cap T)$. Finally, define

$$\sigma(vw) = (c_1(vw), c_2(vw), c_3(vw), c_4(vw)).$$

If n is not of the form $\binom{m}{t}$, then let $n' \geq n$ be the smallest integer of this form, color $\binom{[n']}{2}$ as described above, and restrict the coloring to $\binom{[n]}{2}$. \square

It is known [10, 11] that σ is both a $(3, 2)$ and $(4, 3)$ -coloring of K_n (we only need the first and fourth coordinates of color vectors for this) and, for suitable choice of m and t it uses $e^{O(\sqrt{\log n})}$ colors for all n . We need the following additional properties.

Proposition 2. *The coloring σ satisfies the following properties.*

- 1) If $v < w < x$, then $\sigma(vw) \neq \sigma(wx)$.
- 2) If $v < w < \min\{x, y\}$, and $\sigma(vw) = \sigma(vx)$, then $\sigma(vy) \neq \sigma(wx)$.
- 3) If $v < w < x < y$ with $\sigma(vw) = \sigma(xy)$, then $\sigma(vx) \neq \sigma(vy)$.

Proof. It suffices to consider the first coordinate c_1 of σ to prove the first two properties. For 1), observe that $i = c_1(vw)$ implies that $w(i) = 1$, while $i = c_1(wx)$ implies that $w(i) = 0$. For 2), let $i = c_1(vw) = c_1(vx)$ and suppose for contradiction that $i' = c_1(vy) = c_1(wx)$ so that $v(j) = y(j)$ for $j < i'$. Assume first that $i < i'$. Then $y(i) = v(i) = 0$, while $w(i) = 1$. This implies that $w > y$, a contradiction. Now assume that $i > i'$ ($i = i'$ is impossible since $w(i) = 1$ while $w(i') = 0$). Then $0 = v(i') = x(i') = 1$ due to $c_1(vy) = i', c_1(vx) = i > i'$ and $c_1(wx) = i'$.

We now prove 3) so assume we are given $v < w < x < y$ with $c_1(vw) = c_1(xy) = i < j = c_2(vw) = c_2(xy)$. Then $v(j) = x(j) = 1$ and $y(j) = 0$. Suppose that v, w, x, y are characteristic vectors of V, W, X, Y respectively. Then $c_3(vx) = c_3(VX) = f_V(V \cap X)$ while $c_3(vy) = c_3(VY) = f_V(V \cap Y)$.

If $c_3(vx) = c_3(vy)$, then $f_V(V \cap X) = f_V(V \cap Y)$ and since f_V is a bijection, $V \cap X = V \cap Y$. But this is impossible as $j \in (V \cap X) \setminus Y$. \square

We are now ready to describe the edge-coloring χ of K_n^3 that we will use.

Construction of χ : Given a copy of K_n on $[n]$ and the edge-coloring σ , we produce an edge-coloring χ of the 3-graph H on $\{0, 1\}^n$ as follows. Order the vertices of H according to the integer that they represent in binary. Given vertices $x < y$ in $V(H)$, let γ_{xy} be the first coordinate where x and y differ. Given vertices $x < y < z$, let δ_{xyz} equal 1 if $\gamma_{xy} < \gamma_{yz}$ and -1 otherwise. For an edge uvw with $u < v < w$, let

$$\chi(uvw) = (\sigma(\gamma_{uv}\gamma_{vw}), \delta_{uvw}). \quad \square$$

Since σ is an edge-coloring of K_n with $e^{O(\sqrt{\log n})}$ colors, χ is an edge-coloring of K_N^3 ($N = 2^n$) with $e^{O(\sqrt{\log \log N})}$ colors as promised. Moreover, extending this construction to all N is trivial by considering the smallest $N' \geq N$ which is a power of 2, coloring $\binom{[N']}{2}$ and restricting to $\binom{[N]}{2}$. We are left with showing that χ is a $(5, 3)$ -coloring of K_N^3 .

Proof that χ is a $(5, 3)$ -coloring: Suppose, for contradiction, that $X = \{x_1, \dots, x_5\}$ where $x_1 < x_2 < x_3 < x_4 < x_5$ are five vertices of H forming a 2-colored K_5^3 . Let $\gamma_i = \gamma_{x_i x_{i+1}}$. Let $\gamma = \min \gamma_j$ and assume this minimum is achieved by γ_p . Note that this minimum is uniquely achieved, and $\gamma_i \neq \gamma_{i+1}$ for all i .

Case 1: $p \in \{1, 4\}$. The arguments for both cases are almost identical so we only consider the case $p = 1$. By assumption we have $\gamma_1 < \gamma_2$. First assume that $\gamma_3 > \gamma_2$. If $\gamma_4 > \gamma_3$, then the K_4 on $\{\gamma_i : i \in [4]\}$ has three colors since σ is a $(4, 3)$ -coloring and this gives at least three colors to the edges in X . If $\gamma_4 < \gamma_3$ then the K_3 on $\{\gamma_i : i \in [3]\}$ has two colors since σ is a $(3, 2)$ -coloring and this gives two colors to the edges of H within $\{x_i : i \in [4]\}$ with positive δ -coordinate. On the other hand $\delta_{x_3 x_4 x_5} = -1$, so we again have three colors on X . We now suppose that $\gamma_3 < \gamma_2$. If $\gamma_4 < \gamma_3$, then the K_3 on $\{\gamma_2, \gamma_3, \gamma_4\}$ has two colors since σ is a $(3, 2)$ -coloring and this gives two colors to the edges of H within $\{x_2, x_3, x_4, x_5\}$ with negative δ -coordinate. On the other hand $\delta_{x_1 x_2 x_3} = 1$, so we again have three colors on X . Finally, we may assume that $\gamma_1 < \gamma_3 < \min\{\gamma_2, \gamma_4\}$. Now $\sigma(\gamma_1 \gamma_3) \neq \sigma(\gamma_3, \gamma_4)$ due to property 1) of σ , hence $\chi(x_1 x_3 x_4) \neq \chi(x_3 x_4 x_5)$ and both have positive δ -coordinates. But $\delta_{x_2 x_3 x_4} = -1$, so $\chi(x_2 x_3 x_4)$ is the third color on X .

Case 2: $p \in \{2, 3\}$. The arguments for both cases are almost identical so we only consider the case $p = 2$. We have $\gamma_3 > \gamma_2$. If in addition $\gamma_4 > \gamma_3$, then we get two colors among $\{x_2, x_3, x_4, x_5\}$ with positive δ -coordinate while $\delta_{x_1 x_2 x_3} = -1$. So we may assume that $\gamma_2 < \gamma_4 < \gamma_3$. Now $\chi(x_2 x_3 x_4)$ and $\chi(x_2 x_4 x_5)$ both have positive δ coordinates while $\delta_{x_3 x_4 x_5} = -1$. Hence we have three colors unless $\sigma(\gamma_2 \gamma_3) = \sigma(\gamma_2 \gamma_4)$ which we may assume. Certainly $\delta_{x_1 x_2 x_3} = -1$, so we are done unless $\sigma(\gamma_2 \gamma_1) = \sigma(\gamma_4 \gamma_3)$ which we also assume. If $\gamma_1 = \gamma_4$, then $\sigma(\gamma_2 \gamma_4) = \sigma(\gamma_4 \gamma_3)$ and hence $\{\gamma_2, \gamma_4, \gamma_3\}$ is a monochromatic triangle, contradiction. If $\gamma_1 > \gamma_4$, then $\gamma_2 < \gamma_4 < \min\{\gamma_1, \gamma_3\}$ with $\sigma(\gamma_2 \gamma_4) = \sigma(\gamma_2 \gamma_3)$ and $\sigma(\gamma_2 \gamma_1) = \sigma(\gamma_4 \gamma_3)$. This contradicts property 2). If $\gamma_1 < \gamma_4$, then $\gamma_2 < \gamma_1 < \gamma_4 < \gamma_3$ with $\sigma(\gamma_2 \gamma_1) = \sigma(\gamma_4 \gamma_3)$ and $\sigma(\gamma_2 \gamma_4) = \sigma(\gamma_2 \gamma_3)$. This contradicts property 3) and completes the proof. \square

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References

- [1] D. Conlon, J. Fox, C. Lee, B. Sudakov, On the grid Ramsey problem and related questions, *Int. Math. Res. Not.*, to appear.
- [2] D. Conlon, J. Fox, C. Lee and B. Sudakov, The Erdős-Gyárfás problem on generalized Ramsey numbers, *Proc. London Math. Soc.*, to appear.
- [3] P. Erdős, Problems and results on finite and infinite graphs, in *Recent advances in graph theory* (Proc. Second Czechoslovak Sympos., Prague, 1974), 183–192, Academia, Prague, 1975.
- [4] P. Erdős, Solved and unsolved problems in combinatorics and combinatorial number theory, in *Proceedings of the twelfth southeastern conference on combinatorics, graph theory and computing*, Vol. I (Baton Rouge, La., 1981), *Congr. Numer.* 32 (1981), 49–62.
- [5] P. Erdős and A. Gyárfás, A variant of the classical Ramsey problem, *Combinatorica* 17 (1997), 459–467.
- [6] D. Eichhorn and D. Mubayi, Edge-coloring cliques with many colors on subcliques, *Combinatorica* 20 (2000), 441–444.
- [7] J. Fox and B. Sudakov, Ramsey-type problem for an almost monochromatic K_4 , *SIAM J. Discrete Math.* 23 (2008), 155–162.
- [8] R. L. Graham, B. L. Rothschild and J. H. Spencer, *Ramsey theory*, second ed., Wiley Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons Inc., New York, 1990.
- [9] A. Kostochka and D. Mubayi, When is an almost monochromatic K_4 guaranteed?, *Combin. Probab. Comput.* 17 (2008), 823–830.
- [10] D. Mubayi, Edge-coloring cliques with three colors on all 4-cliques, *Combinatorica* 18 (1998), 293–296.
- [11] D. Mubayi, An explicit construction for a Ramsey problem, *Combinatorica*, 24 (2004), no. 2, 313–324.
- [12] S. Shelah, Primitive recursive bounds for van der Waerden numbers, *J. Amer. Math. Soc.* 1 (1989), 683–697.